Roman Bondage Numbers of Some Graphs*

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Abstract: A Roman dominating function on a graph $G = (V, E)$ is a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u with $f(u) = 0$ is adjacent to at least one vertex v with $f(v) = 2$. The weight of a Roman dominating function is the value $f(G) = \sum_{u\in V} f(u)$. The Roman domination number of G is the minimum weight of a Roman dominating function on G. The Roman bondage number of a nonempty graph G is the minimum number of edges whose removal results in a graph with the Roman domination number larger than that of G. This paper determines the exact value of the Roman bondage numbers of two classes of graphs, complete t-partite graphs and $(n - 3)$-regular graphs with order n for any $n \geq 5$.

Keywords: Combinatorics, Roman domination number, Roman bondage number, t-partite graph, regular graph.
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1 Introduction

In this paper, a graph $G = (V, E)$ is considered as an undirected graph without loops and multi-edges, where $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set. For each vertex $x \in V(G)$, let $N_G(x) = \{y \in V(G) : xy \in E(G)\}$, $N_G[x] = N_G(x) \cup \{x\}$, and $E_G(x) = \{xy : y \in N_G(x)\}$. The cardinality $|E_G(x)|$ is the degree of x, denoted by $d_G(x)$. For two disjoint nonempty and proper subsets S and T in $V(G)$, we use $E_G(S, T)$ to denote the set of edges between S and T in G, and $G[S]$ to denote a subgraph of G induced by S.

A subset $D \subseteq V$ is a *dominating set* of G if $N_G(x) \cap D \neq \emptyset$ for every vertex x in $G-D$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of all dominating sets of G. The *Roman dominating function* on G, proposed by Cockayne *et al.* [1], is a function $f : V \to \{0, 1, 2\}$ such that each vertex x with $f(x) = 0$ is adjacent to at least one vertex y with $f(y) = 2$. For $S \subseteq V$ let $f(S) = \sum\limits_{u\in S} f(u)$. The value $f(V(G))$

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is called the *weight* of f, denoted by $f(G)$. The *Roman domination number*, denoted by $\gamma_{\rm R}(G)$, is defined as the minimum weight of all Roman dominating functions, that is,

$$\gamma_{\rm R}(G) = \min\{f(G) : f \text{ is a Roman dominating function on } G\}.$$

A Roman dominating function f is called a $\gamma_{\rm R}$-*function* if $f(G) = \gamma_{\rm R}(G)$. Roman domination numbers have been studied, see, for example [1, 2, 4–9, 11, 12, 14].

To measure the vulnerability or the stability of the domination in an interconnection network under edge failure, Fink et at. [3] proposed the concept of the bondage number in 1990. The *bondage number*, denoted by $b(G)$, of G is the minimum number of edges whose removal from G results in a graph with larger domination number of G.

The *Roman bondage number*, denoted by $b_{\rm R}(G)$ and proposed first by Rad and Volkmann [10], of a nonempty graph G is the minimum number of edges whose removal from G results in a graph with larger Roman domination number. Precisely speaking, the Roman bondage number

$$b_{\rm R}(G) = \min\{|B| : B \subseteq E(G), \gamma_{\rm R}(G - B) > \gamma_{\rm R}(G)\}.$$

An edge set B that $\gamma_{\rm R}(G - B) > \gamma_{\rm R}(G)$ is called the *Roman bondage set* and the minimum one the *minimum Roman bondage set*. In [6], the authors showed that the decision problem for $b_{\rm R}(G)$ is NP-hard even for bipartite graphs.

For a complete t-partite graph $K_{m_1,m_2,\ldots,m_t}$, its bondage number determined by Fink *et al.* [3] for the undirected case and by Zhang *et al.* [15] for the directed case. Motivated by these results, we, in this paper, consider its Roman bondage number. For a complete t-partite undirected graph $K_{m_1,m_2,\ldots,m_t}$ with $m_1 = m_2 = \ldots = m_i < m_{i+1} \le \ldots \le m_t$ and $n = \sum\limits_{j=1}^{t} m_j$, we determine that

$$b_{\rm R}(K_{m_1,m_2,\ldots,m_t}) = \begin{cases} \lfloor \frac{i}{2} \rfloor & \text{if } m_i = 1 \text{ and } n \ge 3; \\ 2 & \text{if } m_i = 2 \text{ and } i = 1; \\ i & \text{if } m_i = 2 \text{ and } i \ge 2; \\ n-1 & \text{if } m_i = 3 \text{ and } i = t \ge 3; \\ n - m_t & \text{if } m_i \ge 3 \text{ and } m_t \ge 4. \end{cases}$$

Consider $K_{3,3,\ldots,3}$ of order $n \ge 9$, which is an $(n-3)$-regular graph. The above result means that $b_{\rm R}(K_{3,3,\ldots,3}) = n-1$. In this paper, we further determine that $b_{\rm R}(G) = n-2$ for any $(n-3)$-regular graph G of order $n \ge 5$ and $G \ne K_{3,3,\ldots,3}$.

In the proofs of our results, when a Roman dominating function of a graph is constructed, we only give its nonzero value of some vertices.

For terminology and notation on graph theory not given here, the reader is referred to Xu [13].

2 Preliminary results

Lemma 2.1 (Cockayne et al. [1]) *For a complete t-partite graph $K_{m_1,m_2,\ldots,m_t}$ with $1 \le m_1 \le m_2 \le \ldots \le m_t$ and $t \ge 2$,*

$$\gamma_{\rm R}(K_{m_1,m_2,\ldots,m_t}) = \begin{cases} 2, & \text{if } m_1 = 1; \\ 3, & \text{if } m_1 = 2; \\ 4, & \text{if } m_1 \ge 3. \end{cases}$$

Lemma 2.2 (Hu and Xu [6]) *Let G be a graph of order $n \geq 3$ and t be the number of vertices of degree $n-1$ in G. If $t \geq 1$, then $b_{\rm R}(G) = \lceil \frac{t}{2} \rceil$.*

Lemma 2.3 (Hu and Xu [6]) *Let G be a nonempty graph of order $n \geq 3$, then $\gamma_{\rm R}(G) = 3$ if and only if $\Delta(G) = n-2$.*

Lemma 2.4 *Let G be an $(n-3)$-regular graph of order $n \geq 4$. Then $\gamma_{\rm R}(G) = 4$.*

Proof. Since G is an $(n-3)$-regular graph and $n \geq 4$, G is nonempty. Let f be a minimum Roman dominating function of G. If there is no vertex x such that $f(x) = 2$, then $f(y) = 1$ for every vertex y. Therefore, $\gamma_{\rm R}(G) = f(G) \geq 4$. Assume that there is some vertex x with $f(x) = 2$. Let u and v be the only two vertices not adjacent to x in G. If $f(u) = 0$ or $f(v) = 0$, then there exists a vertex $y \neq x$ adjacent to u or v in G such that $f(y) = 2$ and hence $\gamma_{\rm R}(G) = f(G) \geq f(x) + f(y) = 4$. If $f(u) \geq 1$ and $f(v) \geq 1$, then $\gamma_{\rm R}(G) = f(G) \geq f(x) + f(u) + f(v) \geq 4$. In the following, we prove $\gamma_{\rm R}(G) \leq 4$.

For any vertex x, let y and z be the only two vertices not adjacent to x in G. Denote $f(x) = 2$ and $f(y) = f(z) = 1$. Then, f is a Roman dominating function of G with $f(G) = 4$, and hence $\gamma_{\rm R}(G) \leq 4$. Thus, $\gamma_{\rm R}(G) = 4$. ∎

Lemma 2.5 *Let G be an $(n-3)$-regular graph of order $n \geq 5$ and B be a Roman bondage set of G. Then $E_G(x) \cap B \neq \emptyset$ for any $x \in V(G)$.*

Proof. By Lemma 2.4, $\gamma_{\rm R}(G) = 4$. Let $G' = G - B$. Then $\gamma_{\rm R}(G') > 4$ since B is a Roman bondage set in G. By contradiction, assume $E_G(x) \cap B = \emptyset$ for some $x \in V(G)$. Let y and z be the only two vertices not adjacent to x in G. Denote $f(x) = 2$ and $f(y) = f(z) = 1$. Since every $u \notin \{x, y, z\}$ is adjacent to x, f is a Roman dominating function of G' with $f(G') = 4$. Thus we obtain a contradiction as follows. $\gamma_{\rm R}(G') \leq f(G') = 4 < \gamma_{\rm R}(G')$. ∎

Lemma 2.6 *Let G be an $(n-3)$-regular graph of order $n \geq 5$ and B be a Roman bondage set of G, x be any vertex, y and z be the only two vertices not adjacent to x in G. If $E_G(x) \cap B = \{xw\}$, then $|E_G(\{y, z, w\}, x') \cap B| \geq 1$ for any vertex $x' \in V(G) \setminus \{x, y, z, w\}$ that is adjacent to each vertex in $\{y, z, w\}$ in G.*

Proof. Since $E_G(x) \cap B \neq \emptyset$ by Lemma 2.5, there is a vertex w such that $xw \in E_G(x) \cap B$. Let $G' = G - B$. Then $\gamma_{\rm R}(G') > 4$ by Lemma 2.4. By contradiction, suppose $E_G(\{y, z, w\}, x') \cap B = \emptyset$ for some vertex $x' \in V(G) \setminus \{x, y, z, w\}$ that is adjacent to each in $\{y, z, w\}$ in G. Set $f(x) = f(x') = 2$. Then, f is a Roman dominating function of G' with $f(G') = 4$ since $N_{G'}[x] \cup N_{G'}[x'] = V(G)$, a contradiction. ∎

Lemma 2.7 *Let G be an $(n-3)$-regular graph of order $n \geq 6$ and B be a Roman bondage set of G. For three vertices x, y and z that are pairwise non-adjacent in G, if each of them is incident with exact one edge in B, then $|B| \geq n-2$ and, moreover, $|B| \geq n-1$ if $G = K_{3,3,\ldots,3}$.*

Proof. By the hypothesis, for any $v \in \{x, y, z\}$, $|E_G(v) \cap B| = 1$ and v is adjacent to every vertex in $V(G - \{x, y, z, v\})$. Let $xu \in E_G(x) \cap B$. We claim $yu \in E_G(y) \cap B$ and $zu \in E_G(z) \cap B$. In fact, by contradiction, without loss of generality suppose $yv \in E_G(y) \cap B$ and $zw \in E_G(z) \cap B$ with $u \neq v$ and $u \neq w$. Then, u is adjacent to y

and z in $G-B$. Set $f(x)=f(u)=2$. Then f is a Roman dominating function of G with $f(G-B)=4$, which contradicts with $\gamma_R(G-B)>4$ by Lemma 2.4.

Let s and t be the only two vertices not adjacent to u in G, and let $V'=V(G)\setminus\{x,y,z,u,s,t\}$. By the hypothesis, each vertex in $\{y,z,u\}$ is adjacent to all vertices in V' in G. By Lemma 2.6, for any vertex $x'\in V'$, if such a vertex exists, $|E_G(\{u,y,z\},x')\cap B|\geq 1$, and so

$$|E_G(\{u,y,z\},V')\cap B|\geq |V'|=n-6. \tag{2.1}$$

By Lemma 2.5, $|E_G(s)\cap B|\geq 1$ and $|E_G(t)\cap B|\geq 1$, and so we have that

$$|(E_G(s)\cup E_G(t))\cap B| \left\{\begin{array}{ll} \geq 1 & \text{if} \;\; st\in E(G); \\ =2 & \text{if} \;\; st\notin E(G). \end{array}\right. \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$\begin{array}{rl} |B| & \geq |\{xu,yu,zu\}|+|(E_G(s)\cup E_G(t))\cap B| \\ & \quad +|E_G(\{u,y,z\},V')\cap B| \\ & \geq \left\{\begin{array}{ll} n-2 & \text{if} \;\; st\in E(G); \\ n-1 & \text{if} \;\; st\notin E(G). \end{array}\right. \end{array}$$

If $G=K_{3,3,\ldots,3}$, then $|(E_G(s)\cup E_G(t))\cap B|\geq 2$ since then $st\notin E(G)$ and, hence, $|B|\geq n-1$. ∎

Lemma 2.8 *Let G be an $(n-3)$-regular graph of order $n\geq 5$ and B be a Roman bondage set of G. Let $x\in V(G)$, y and z be the only two vertices not adjacent to x in G. Let $E_G(x)\cap B=\{xw\}$ and $G'=G-B$. Then $|E(G'[\{y,z,w\}])|\leq 1$, that is,*

$$|E(G[\{y,z,w\}])\cap B|\geq \left\{\begin{array}{ll} 1 & \text{if} \;\; |E(G[\{y,z,w\}])|=2; \\ 2 & \text{if} \;\; |E(G[\{y,z,w\}])|=3. \end{array}\right.$$

Proof. Suppose to the contrary that $|E(G'[\{y,z,w\}])|\geq 2$. Without loss of generality, let $yw,zw\in E(G')$. Denote $f(x)=f(w)=2$. Note that x is adjacent to every vertex except w, y and z in G'. Thus, f is a Roman dominating function of G' with $f(G')=4$, a contradiction with $\gamma_{\rm R}(G')>4$ by Lemma 2.4. ∎

3 Results on complete t-partite graphs

Theorem 3.1 *Let $G=K_{m_1,m_2,\ldots,m_t}$ be a complete t-partite graph with $m_1=m_2=\ldots=m_i<m_{i+1}\leq\ldots\leq m_t$, $t\geq 2$ and $n=\sum\limits_{j=1}^{t} m_j$. Then*

$$b_{\rm R}(G)=\left\{\begin{array}{ll} \lceil\frac{i}{2}\rceil & \text{if } m_i=1 \text{ and } n\geq 3; \\ 2 & \text{if } m_i=2 \text{ and } i=1; \\ i & \text{if } m_i=2 \text{ and } i\geq 2; \\ n-1 & \text{if } m_i=3 \text{ and } i=t\geq 3; \\ n-m_t & \text{if } m_i\geq 3 \text{ and } m_t\geq 4. \end{array}\right.$$

Proof. Let $\{X_1, X_2, \ldots, X_t\}$ be the corresponding t-partitions of $V(G)$.

(1) If $m_i = 1$ and $n \geq 3$, then G has i vertices of degree $n-1$, and so $b_{\rm R}(G) = \lceil \frac{i}{2} \rceil$ by Lemma 2.2.

(2) If $m_i = 2$, then $\Delta(G) = n-2$. By Lemma 2.1, $\gamma_{\rm R}(G) = 3$. Let $B \subseteq E(G)$ be a Roman bondage set of G with $|B| = b_{\rm R}(G)$ and $G' = G - B$. Then $\gamma_{\rm R}(G') > \gamma_{\rm R}(G) = 3$, and so $\Delta(G') \leq n-3$ by Lemma 2.3. Thus, $|B \cap E_G(x)| \geq 1$ for every vertex in X_j $(1 \leq j \leq i)$, that is, $|B| \geq 2$ if $i = 1$ and $|B| \geq i$ if $i > 1$.

If $i = 1$, then only two vertices of degree $n-2$ are in X_1, and the removal of any two edges incident with distinct vertices in X_1 results in a graph G'' with $\Delta(G'') \leq n-3$, and hence $\gamma_{\rm R}(G'') \neq 3$ by Lemma 2.3. Since $\gamma_{\rm R}(G'') \geq \gamma_{\rm R}(G) = 3$, $\gamma_{\rm R}(G'') \geq 4$. Thus, $b_{\rm R}(G) \leq 2$ and hence $b_{\rm R}(G) = 2$.

If $i > 1$, then the subgraph H induced by $\bigcup\limits_{j=1}^{i} X_j$ of G is a complete i-partite graph with each partition consisting of two vertices, which is 2-edge-connected and $2(i-1)$-regular, and so has a perfect matching M with $|M| = i$. Thus, $G - M$ has the maximum degree $n-3$ and hence $\gamma_{\rm R}(G-M) \neq 3$ by Lemma 2.3. Since $\gamma_{\rm R}(G-M) \geq \gamma_{\rm R}(G) = 3$, $\gamma_{\rm R}(G-M) \geq 4$. Thus, $b_{\rm R}(G) \leq |M| = i$, and so $b_{\rm R}(G) = i$.

(3) Assume $m_i = 3$ and $i = t$. Then G is $(n-3)$-regular. Note that if $t = 2$ then $n = 6$ and $b_R(K_{3,3}) = 4 = n-2$. It is easy to verify that the conclusion is true for $t = 3, 4$, and assume $t \geq 5$ below. Let $x \in V(G)$ and $H = G - E_G(x)$, then $\gamma_{\rm R}(H) = 1 + \gamma_R(K_{2,3,\ldots,3}) = 4$ by Lemma 2.1. By the conclusion (2) just showed, $b_{\rm R}(K_{2,3,\ldots,3}) = 2$ and hence

$$b_{\rm R}(G) \leq |E_G(x)| + b_{\rm R}(K_{2,3,\ldots,3}) = (n-3) + 2 = n-1.$$

We now prove that $b_{\rm R}(G) \geq n-1$. By contradiction. Assume that there is a Roman bondage set B of G such that $|B| \leq n-2$. Let $G' = G - B$. Then $\gamma_{\rm R}(G') > \gamma_{\rm R}(G) = 4$ by Lemma 2.1, and $|E_G(x) \cap B| \geq 1$ for any vertex $x \in V(G)$ by Lemma 2.5. If $|E_G(x) \cap B| \geq 2$ for any vertex $x \in V(G)$, then the subgraph induced by B has the minimum degree at least two, and so $|B| \geq n$, a contradiction. Thus, there is a vertex x_1 in G such that $|E_G(x_1) \cap B| = 1$. Let $x_1y_1 \in B$ and, without loss of generality, let $X_1 = \{x_1, x_2, x_3\}$ and $X_2 = \{y_1, y_2, y_3\}$. By Lemma 2.8,

$$|E(G[\{y_1, x_2, x_3\}]) \cap B| \geq 1, \tag{3.1}$$

and by Lemma 2.5,

$$|E_G(y_2) \cap B| \geq 1 \ \text{ and } |E_G(y_3) \cap B| \geq 1. \tag{3.2}$$

Let $V_1 = V(G) \setminus (X_1 \cup X_2)$. By Lemma 2.6,

$$|E_G(\{y_1, x_2, x_3\}, x') \cap B| \geq 1 \ \text{ for any } x' \in V_1, \tag{3.3}$$

and so

$$|E_G(\{y_1, x_2, x_3\}, V_1) \cap B| \geq n-6. \tag{3.4}$$

It follows from (3.1), (3.2) and (3.4) that

$$\begin{aligned} n-2 \geq |B| \ & \geq |\{x_1y_1\}| + |E(G[\{y_1, x_2, x_3\}]) \cap B| \\ & \quad\ + |E_G(\{y_1, x_2, x_3\}, V_1) \cap B| + |E_G(y_2) \cap B| \\ & \quad\ + |E_G(y_3) \cap B| + |E(G[V_1]) \cap B| \\ & \geq 1 + 1 + (n-6) + 1 + 1 + 0 \\ & \geq n-2. \end{aligned} \tag{3.5}$$

Thus, all the equalities in (3.5) hold, which implies that all the equalities in (3.1), (3.2) and (3.3) hold, and $|E(G[V_1]) \cap B| = 0$.

Let $E_G(y_2) \cap B = \{y_2u\}$ and $E_G(y_3) \cap B = \{y_2v\}$. The worst case is that u and v belong to different partitions of $X_3, \ldots, X_t$. Since $t \geq 5$, there exists some i with $3 \leq i \leq t$ such that both u and v are not belong to X_i. Thus, each vertex in X_i is incident with exact one edge in B. By Lemma 2.7, $|B| \geq n - 1$, a contradiction.

Thus, $b_{\rm R}(K_{3,3,\ldots,3}) = n - 1$.

(4) We now assume $m_i \geq 3$ and $m_t \geq 4$. By Lemma 2.1, we have $\gamma_{\rm R}(G) = 4$. Let u be a vertex in X_t and f be a $\gamma_{\rm R}$-function of $G - E_G(u)$. Then u is an isolated vertex. Thus, $f(u) = 1$ and $f(G - u) = 4$ by Lemma 2.1 since $G - u$ is a complete t-partite graph with at least 3 vertices in every partition. Thus $\gamma_{\rm R}(G - E_G(u)) = 5 > 4 = \gamma_{\rm R}(G)$, and hence $b_{\rm R}(G) \leq |E_G(u)| = n - m_t$.

We now show $b_{\rm R}(G) \geq n - m_t$. Let B be a minimum Roman bondage set of G, and let $G' = G - B$. Then $\gamma_{\rm R}(G') > \gamma_{\rm R}(G) = 4$.

Assume that there is a vertex x in G such that $E_G(x) \cap B = \emptyset$. Then there is some j with $1 \leq j \leq t$ such that $x \in X_j$. If there exists some $y \in V(G - X_j)$ such that $E_G(y, X_j) \cap B = \emptyset$. Set $f(x) = f(y) = 2$. Then f is a Roman dominating function of G' with $f(G') = 4$, a contradiction. Thus,

$$E_G(y, X_j) \cap B \neq \emptyset \ \text{ for any } y \in V(G - X_j).$$

It follows that

$$|B| \geq |V(G) \setminus X_j| = n - m_j \geq n - m_t.$$

We now assume that

$$|E_G(x) \cap B| \geq 1 \ \text{ for any } x \in V(G). \tag{3.6}$$

If $|E_G(x) \cap B| \geq 2$ for any $x \in V(G)$, then the subgraph induced by B has the minimum degree at least two, from which we have $|B| \geq n > n - m_t$.

We suppose that there exists a vertex $x_1 \in V(G)$ such that $|E_G(x_1) \cap B| = 1$. Let $x_1 \in X_j$ and $x_2, x_3, \ldots, x_{m_j}$ be the other vertices of X_j. Let y_1 be the unique neighbor of x_1 in $E_G(x_1) \cap B$, and let X_k contains y_1. Let $V' = V(G) \setminus (X_j \cup X_k)$ and $V'' = \{y_1, x_2, x_3, \ldots, x_{m_j}\}$. If there is some $x' \in V'$ such that $|E_G(x', V'') \cap B| = 0$, set $f(x) = f(x') = 2$, then f is a Roman dominating function of G' with $f(G') = 4$, a contradiction. Thus,

$$|E_G(x', V'') \cap B| \geq 1 \ \text{ for any } x' \in V'. \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$b_{\rm R}(G) = |B| \geq |V'| + |X_k| \geq n - m_t.$$

Thus, $b_{\rm R}(G) = n - m_t$.

The theorem follows. ∎

4 Results on $(n-3)$-regular graphs

By Theorem 3.1, we immediately have $b_{\mathrm{R}}(K_{3,3,\ldots,3}) = n-1$ if its order is n. $K_{3,3,\ldots,3}$ is an $(n-3)$-regular graph of order n. In this section, we determine the Roman bondage number of any $(n-3)$-regular graph G of order n is equal to $n-2$ if $G \neq K_{3,3,\ldots,3}$.

Lemma 4.1 *Let G be an $(n-3)$-regular graph of order $n \geq 7$ but $G \neq K_{3,3,\ldots,3}$ and B be a Roman bondage set of G. Let $x, w \in V(G)$ and $xw \in E(G)$. Let y, z and p, q be the only two vertices not adjacent to x and w in G, respectively. If $E_G(x) \cap B = \{xw\}$ and $\{y, z\} \cap \{p, q\} \neq \emptyset$, then $|B| \geq n-2$.*

Proof. By Lemma 2.4, $\gamma_{\mathrm{R}}(G) = 4$. Let $G' = G - B$. Then $\gamma_{\mathrm{R}}(G') > 4$. By Lemma 2.5, $E_G(y') \cap B \neq \emptyset$ for any $y' \in V(G)$. By contradiction, assume $|B| \leq n-3$. We now deduce a contradiction by considering the following two cases.

Case 1 $\{y, z\} = \{p, q\}$.

In this case, $yz \in E(G)$ since G is $(n-3)$-regular. Let $U_1 = V(G) \setminus \{x, y, z, w\}$. Then any vertex in U_1 is adjacent to each in $\{w, y, z\}$. By Lemma 2.6, we have that $|E_G(\{w, y, z\}, x') \cap B| \geq 1$, and so $|E_G(\{w, y, z\}, U_1) \cap B| \geq |U_1| = n-4$. It follows that

$$\begin{aligned} n-3 \geq |B| &\geq |\{xw\}| + |E_G(\{w, y, z\}, U_1) \cap B| + |E(G[U_1]) \cap B| \\ &\geq 1 + (n-4) + 0 \\ &= n-3. \end{aligned} \tag{4.1}$$

This means that all equalities in (4.1) hold, that is, $yz \notin B$, $E(G[U_1]) \cap B = \emptyset$, $|E_G(\{w, y, z\}, x') \cap B| = 1$ and then, $|E_G(x') \cap B| = 1$ for any vertex $x' \in U_1$. Let $yr \in B$ for some $r \in U_1$ since $E_G(y) \cap B \neq \emptyset$, s and t be the only two vertices not adjacent to r in G.

Assume $st \notin E(G)$. Then r, s, t be three vertices not adjacent to each other in G, and each one of them is incident with exact one edge in B. By Lemma 2.7, $|B| \geq n-2$, a contradiction.

Now, assume $st \in E(G)$. We claim that $ys \in B$ and $yt \in B$. By contradiction, assume $ys \notin B$. Denote $f(r) = f(s) = 2$. Then, f is a Roman dominating function of G' with $f(G') = 4$, a contradiction. Also, $yt \in B$ by replace t by s. Then zs and zt not belong to B. Denote $f(r) = f(z) = 2$. Then, f is a Roman dominating function of G' with $f(G') = 4$, a contradiction.

Case 2 $|\{y, z\} \cap \{p, q\}| = 1$. Without loss of generality, let $p = y$.

In this case, $yz, wz \in E(G)$ and hence $|E(G[\{y, z, w\}]) \cap B| \geq 1$ by Lemma 2.8. Let r be the only vertex except x not adjacent to z in G. By Lemma 2.6, $|E_G(\{w, y, z\}, x') \cap B| \geq 1$ for any vertex $x' \in U_2 = V(G) \setminus \{x, y, z, w, q, r\}$.

If $q = r$, then $|E_G(\{w, y, z\}, U_2) \cap B| \geq |U_2| = n-5$. Then we can deduce a contradiction as follows.

$$\begin{aligned} n-3 \geq |B| &\geq |\{xw\}| + |E_G(\{w, y, z\}, U_2) \cap B| \\ &\quad + E(G[\{y, z, w\}]) \cap B| + |E_G(q) \cap B| \\ &\geq 1 + (n-5) + 1 + 1 \\ &= n-2. \end{aligned}$$

If $q \neq r$, then $wr, zq \in E(G)$ and $|E_G(\{w,y,z\},U_2) \cap B| \geq |U_2| = n-6$. Then,

$$\begin{aligned} n-3 \geq |B| \; & \geq |\{xw\}| + |E_G(\{w,y,z\},U_2) \cap B| + |E(G[U_2]) \cap B| \\ & \quad + E(G[\{y,z,w\}]) \cap B| + |(E_G(q) \cup E_G(r)) \cap B| \\ & \geq 1 + (n-6) + 0 + 1 + 1 \\ & = n-3. \end{aligned} \tag{4.2}$$

It follows that the equalities in (4.2) hold, which implies that $|(E_G(q) \cup E_G(r)) \cap B| = 1$, $E(G[U_2]) \cap B = \emptyset$, $|E_G(\{w,y,z\},x') \cap B| = 1$ and then, $|E_G(x') \cap B| = 1$ for any vertex $x' \in U_2$. Then $(E_G(q) \cup E_G(r)) \cap B = \{qr\}$, and hence $wr \notin B$, $zq \notin B$.

Let s be the only vertex except w not adjacent to q in G. Then both rs and ws not belong to G', otherwise denote $f(q) = f(r) = 2$ or $f(q) = f(w) = 2$. Then f is a Roman dominating function of G' with $f(G') = 4$, a contradiction. $rs, ws \notin E(G')$ imply that $ws \in B$ and $rs \notin E(G)$. Then $zs \in E(G)$ and $zs \notin B$ since $|E_G(\{w,y,z\},s) \cap B| = 1$. Denote $f(r) = f(z) = 2$. Then f is a Roman dominating function of G' with $f(G') = 4$, a contradiction. Thus, $|B| \geq n-2$.

The lemma follows. ∎

Lemma 4.2 *let G be an $(n-3)$-regular graph of order $n \geq 7$ but $G \neq K_{3,3,\ldots,3}$ and B be a Roman bondage set of G. Let $x, w \in V(G)$ and $xw \in E(G)$. If $E_G(x) \cap B = E_G(w) \cap B = \{xw\}$, then $|B| \geq n-2$.*

Proof. Let y, z and p, q be the only two vertices not adjacent to x and w in G, respectively.

We claim that $\{y,z\} \cap \{p,q\} \neq \emptyset$. By contradiction, suppose $\{y,z\} \cap \{p,q\} = \emptyset$. Then $wy, wz \in E(G)$, and $wy, wz \notin B$ since $E_G(w) \cap B = \{xw\}$. Denote $f(x) = f(w) = 2$. Then f is a Roman dominating function of G' with $f(G') = 4$, a contradiction. Thus $\{y,z\} \cap \{p,q\} \neq \emptyset$, and hence $|B| \geq n-2$ by Lemma 4.1. ∎

Theorem 4.1 *Let G be an $(n-3)$-regular graph of order $n \geq 5$ but $G \neq K_{3,3,\ldots,3}$. Then $b_{\mathrm{R}}(G) = n-2$.*

Proof. We first consider $n \in \{5,6\}$. If $n = 5$, then $G = C_5$, and so $b_{\mathrm{R}}(G) = 3$. If $n = 6$, then G is the Cartesian product of a cycle C_3 and a complete graph K_2, that is, $G = C_3 \times K_2$, and so $b_{\mathrm{R}}(G) = 4$. In the following, suppose $n \geq 7$.

By Lemma 2.4, $\gamma_{\mathrm{R}}(G) = 4$. Let $x_0 \in V(G)$ and $y_0z_0 \in E(G)$, where y_0 and z_0 are the only two vertices not adjacent to x_0 in G. We consider the Roman domination number of $H = G - x_0 - y_0z_0$. Since H is $(|V(H)|-3)$-regular and $|V(H)| \geq 4$, $\gamma_{\mathrm{R}}(H) = 4$ by Lemma 2.4. Thus $\gamma_{\mathrm{R}}(G - E_G(x_0) - y_0z_0) \geq 5$ and hence $b_{\mathrm{R}}(G) \leq |E_G(x_0)| + 1 = n-2$. Next, we prove that $b_{\mathrm{R}}(G) \geq n-2$.

Let B be a minimum Roman bondage set of G and $G' = G - B$. Then $|B| \leq n-2$ and $\gamma_{\mathrm{R}}(G') > 4$. We now prove $|B| \geq n-2$. By contradiction, assume $|B| \leq n-3$. By Lemma 2.5, $E_G(y') \cap B \neq \emptyset$ for any $y' \in V(G)$. Then there exists a vertex x such that $|E_G(x) \cap B| = 1$. Let $xw \in B$, y and z be the only two vertices not adjacent to x in G. Let p and q be the only two vertices not adjacent to w in G. If $\{y,z\} \cap \{p,q\} \neq \emptyset$, then $|B| \geq n-2$ by Lemma 4.1. Thus, we only need to consider the case of $\{y,z\} \cap \{p,q\} = \emptyset$. In this case, $wy, wz \in E(G)$. We now deduce a contradiction by considering the following two cases.

Case 1 $yz \notin E(G)$.

By Lemma 2.8, $|E(G[\{y,z,w\}]) \cap B| \geq 1$. By Lemma 2.6, $|E_G(\{w,y,z\},x') \cap B| \geq 1$ for any vertex $x' \in X_1 = V(G) \setminus \{x,y,z,w,p,q\}$, and so $|E_G(\{w,y,z\},X_1) \cap B| \geq |X_1| = n-6$. Then,

$$\begin{aligned} n-3 \geq |B| &\geq |\{xw\}| + |E_G(\{w,y,z\},X_1) \cap B| \\ &\qquad + |E(G[\{y,z,w\}]) \cap B| + |(E_G(p) \cup E_G(q)) \cap B| \\ &\geq 1 + (n-6) + 1 + 1 \\ &= n-3. \end{aligned} \tag{4.3}$$

It follows that the equalities in (4.3) hold, which implies that $|E_G(\{p,q\}) \cap B| = 1$. Then $(E_G(p) \cup E_G(q)) \cap B = \{pq\}$ and then, $E_G(p) \cap B = E_G(q) \cap B = \{pq\}$. By Lemma 4.2, $|B| \geq n-2$, a contradiction.

Case 2 $yz \in E(G)$.

Let r and s be the only vertex except x not adjacent to y and z in G, respectively. By Lemma 2.8, $|E(G[\{w,y,z\}]) \cap B| \geq 2$. By Lemma 2.6, $|E_G(\{w,y,z\},x') \cap B| \geq 1$ for any vertex $x' \in X_2 = V(G) \setminus \{x,y,z,w,p,q,r,s\}$. Thus, we have that

$$|E_G(\{w,y,z\},X_2) \cap B| \geq |X_2| \geq \begin{cases} n-6 & \text{if } |\{r,s\} \cup \{p,q\}| \leq 2; \\ n-7 & \text{if } |\{r,s\} \cup \{p,q\}| = 3; \\ n-8 & \text{if } |\{r,s\} \cup \{p,q\}| = 4, \end{cases} \tag{4.4}$$

and

$$|(E_G(p) \cup E_G(q) \cup E_G(r) \cup E_G(s)) \cap B| \geq \begin{cases} 1 & \text{if } |\{r,s\} \cup \{p,q\}| \leq 2; \\ 2 & \text{if } |\{r,s\} \cup \{p,q\}| = 3; \\ 2 & \text{if } |\{r,s\} \cup \{p,q\}| = 4, \end{cases} \tag{4.5}$$

It follows from (4.4) and (4.5) that

$$\begin{aligned} n-3 \geq |B| &\geq |\{xw\}| + |E_G(\{w,y,z\},X_2) \cap B| + |E(G[\{w,y,z\}]) \cap B| \\ &\qquad + |(E_G(p) \cup E_G(q) \cup E_G(r) \cup E_G(s)) \cap B| \\ &\geq \begin{cases} n-2 & \text{if } |\{r,s\} \cup \{p,q\}| \leq 3; \\ n-3 & \text{if } |\{r,s\} \cup \{p,q\}| = 4. \end{cases} \end{aligned} \tag{4.6}$$

The Eq. (4.6) implies that $|\{r,s\} \cup \{p,q\}| = 4$, $|B| = n-3$ and $|(E_G(p) \cup E_G(q) \cup E_G(r) \cup E_G(s)) \cap B| = 2$. Then there exist two vertices u,v in $\{p,q,r,s\}$ such that $E_G(u) \cap B = E_G(v) \cap B = \{uv\}$. By Lemma 4.2, $|B| \geq n-2$, a contradiction.

Thus, $b_{\mathrm{R}}(G) = n-2$, and so the theorem follows. ∎

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